

Comments on operators with large spin

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ABSTRACT: We consider high spin operators. We give a general argument for the logarithmic scaling of their anomalous dimensions which is based on the symmetries of the problem. By an analytic continuation we can also see the origin of the double logarithmic divergence in the Sudakov factor. We show that the cusp anomalous dimension is the energy density for a flux configuration of the gauge theory on $AdS_3 \times S^1$. We then focus on operators in $\mathcal{N} = 4$ super Yang Mills which carry large spin and $SO(6)$ charge and show that in a particular limit their properties are described in terms of a bosonic $O(6)$ sigma model. This can be used to make certain all loop computations in the string theory.

KEYWORDS: Supersymmetric gauge theory, AdS-CFT Correspondence, Deep Inelastic Scattering.

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1. Introduction

In this paper we focus on two issues. First we discuss how the so called “cusp anomalous dimension”, $f(\lambda)$, appears in various computations. Namely in the dimension of high spin operators and in lightlike Wilson loops with a cusp. These are well known relations and our only objective here is to give a different perspective on these issues. First we give a general argument for the logarithmic behavior of the anomalous dimension of high spin operators $\Delta - S = f(\lambda) \log S$ [1, 2] which is based on the symmetries of the problem. Then we argue that the Sudakov form factor for two light-like particles has a behavior $e^{-\frac{f(\lambda)}{4}(\log \mu)^2}$ as a function of the IR cutoff [3–6] (for a review see [7]). This factor gives the leading IR behavior when we consider the exclusive scattering of colored particles and it is an important ingredient in the computation of amplitudes [8]. Both of these properties follow from symmetries of the theory plus the fact that we have gauge fluxes.

For the particular case of planar $\mathcal{N} = 4$ super Yang Mills, exact results were derived using integrability. In particular, an exact integral equation was written whose solution gives the cusp anomalous dimension for arbitrary λ [9]. This equation was analyzed further in [10].

In the second part of this paper we consider the question of high spin operators in $\mathcal{N} = 4$ super Yang Mills that carry spin S and one of the $SO(6)$ charges, J , in the large S, J limit such that $J/(\log S)$ is kept finite. In that case one can show that the anomalous dimension continues to have a logarithmic scaling

$$\Delta - S = \left[f(\lambda) + \epsilon \left(\lambda, \frac{J}{\log S} \right) \right] \log S \tag{1.1}$$

This type of operators were studied in [11] (see also [12]), where the function ϵ was computed up to one loop in the strong coupling expansion. We derive an exact expression for ϵ in a suitable limit. We do this by noticing that the full $AdS_5 \times S^5$ sigma model reduces to the $O(6)$ bosonic sigma model in a suitable limit. In the $O(6)$ sigma model we have a configuration with finite charge density whose free energy can be computed as a solution of an integral equation [13]. The decoupling limit that gives the $O(6)$ sigma model involves taking a strong 't Hooft coupling and a small $J/(\log S)$ in such a way that quantum corrections remain finite. Interestingly, the massive excitations of the $O(6)$ sigma model, which are in the vector of $SO(6)$, can be interpreted as insertions of the fundamental scalar fields ϕ^I .

This provides a way to compute higher loop corrections in the string theory side without too much effort. These results could then be compared to a suitable generalization of the BES equation [9] for finite $J/(\log S)$, which will probably be sensitive to higher order corrections of the phase of the fundamental magnon S matrix. (We do not perform this computation here).

This paper is organized as follows. In section two we explain the $\log S$ scaling of high spin operators using the symmetries of a conformal gauge theory. We then discuss the related issue of the double logarithmic infrared divergencies in the Sudakov form factors. We also discuss how to simplify the strong coupling computation of $f(\lambda)$ in $\mathcal{N} = 4$ super Yang Mills at tree level and one loop by using the symmetries we mentioned above. (These computations were originally performed in [14] and [11]).

In section three we discuss high spin operators in $\mathcal{N} = 4$ super Yang Mills and the reduction to an $O(6)$ sigma model.

Finally, in section four we present some conclusions.

2. High spin operators and Sudakov factors in conformal gauge theories

2.1 High spin operators and the $\log S$ scaling

In this section we would like to offer a geometric argument for the logarithmic behavior of the anomalous dimensions of high spin operators in gauge theories. Namely, we consider operators with very high spin $S \rightarrow \infty$ keeping the twist finite. For simplicity, consider first

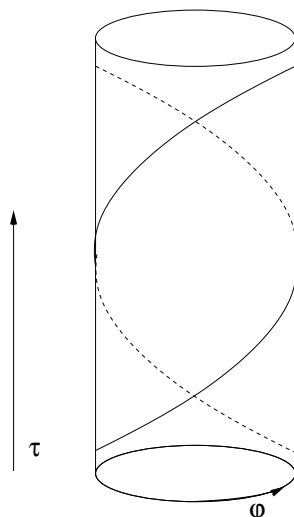


Figure 1: Quark and antiquark moving very fast on opposite sides of the cylinder. They become localized and can be replaced by light-like Wilson lines.

operators of the schematic form

$$\mathcal{O}_S = \bar{q}(\mathcal{D}^{\leftrightarrow})^S q \tag{2.1}$$

where q is in the fundamental representation. These operators have conformal dimensions of the form [1, 2]

$$\Delta - S = \frac{f(\lambda)}{2} \log S \tag{2.2}$$

for large S . The factor of $1/2$ is a convention. Here we are considering a theory with a large number of colors and we are disregarding the mixing between operators with different numbers of quarks. Alternatively, we could be considering the weakly coupled theory in which this mixing is suppressed in perturbation theory for the lowest twist operators.

We can also consider an operator of a similar form in a theory with only adjoint fields, such as $\mathcal{N} = 4$ super Yang Mills. In that case we consider a single trace operator of the schematic form $Tr[\phi^I(\mathcal{D}^{\leftrightarrow})^S \phi^I]$. In the planar limit the high spin anomalous dimension goes as $\Delta - S = f(\lambda) \log S$, which is twice the value we had in (2.2) because, at large N we can view an adjoint particle a quark and an antiquark, each of which gives rise to (2.2).

We now present an argument that explains the $\log S$ behavior in (2.2). A previous argument can be found in [2]. In a conformal field theory, the anomalous dimension of an operator is equal to the energy of the corresponding state of the field theory on the cylinder $R \times S^3$. On the cylinder, a high spin operator consists of two particles (or group of particles) that are moving very rapidly along a great circle of S^3 . These particles are colored and the color field lines go between the two particles. See figure 1.

For simplicity, let us first consider the case of a quark and an antiquark moving very fast along a great circle of the S^3 , with color gauge fields joining them. Parametrizing the cylinder as

$$ds^2_{R \times S^3} = -d\tau^2 + \cos^2 \theta d\varphi^2 + d\theta^2 + \sin^2 \theta d\psi^2 \tag{2.3}$$

we can then imagine that the quark is close to the line $\varphi = \tau$ and the antiquark at $\varphi = \tau + \pi$, both at $\theta = 0$.

Notice that if we had a color neutral object that is moving fast on the sphere, then its energy would go like S , namely $\Delta - S \sim \text{finite}$, since we can get a particle which is moving fast along the equator from a particle that is at rest by applying conformal transformations. In other words from an operator \mathcal{O} , we can consider its descendent $\partial^S \mathcal{O}$. In our case we have a pair of particles and each particle carries color indices. In this case we have a large contribution to $\Delta - S$ from the color electric field lines emanating from the particles. In order to evaluate the effects of these fields, it is convenient to replace the quark and the anti-quark by a Wilson line. Thus, we first consider a Wilson line, which corresponds to $S = \infty$, and then we go back to finite S . We consider a lightlike Wilson line at $\theta = 0$ and $\varphi = \tau$ and an oppositely oriented line along $\varphi = \tau + \pi$, see figure 1. This configuration is clearly invariant under $\tau \rightarrow \tau + c$, $\varphi \rightarrow \varphi + c$ where c is a constant. Less obvious is the fact that these Wilson lines are also invariant under a second symmetry, which acts as a conformal transformation which is not an isometry of $R \times S^3$. As we will see, the $\log S$ behavior is associated to this second symmetry. In order to exhibit this symmetry more clearly we can make a Weyl transformation of the $R \times S^3$ metric (2.3) to $AdS_3 \times S^1$ by writing

$$\begin{aligned}
 ds_{R \times S^3}^2 &= \sin^2 \theta \left[\frac{-d\tau^2 + \cos^2 \theta d\varphi^2 + d\theta^2}{\sin^2 \theta} + d\psi^2 \right] = \sin^2 \theta ds_{AdS_3 \times S^1}^2 \\
 ds_{AdS_3}^2 &= -\cosh^2 \rho d\tau^2 + \sinh^2 \rho d\varphi^2 + d\rho^2, \\
 \sinh \rho &= \frac{1}{\tan \theta}
 \end{aligned} \tag{2.4}$$

A conformal field theory should be Weyl invariant and thus we should get the same result for the Wilson loop if we consider it on $AdS_3 \times S^1$.¹ $AdS_3 \times S^1$ is the space where the field theory is defined and it should not be confused with the AdS_5 space that will appear later when we consider the gravity dual of the field theory. For the moment we are making an argument purely in the context of the field theory and is valid regardless of the value of the coupling or whether the theory has a gravity dual or not. Thus we are considering the four dimensional field theory on a four dimensional space which happens to be $AdS_3 \times S^1$. The Wilson lines, which sit at $\theta = 0$, are mapped to a pair of lines along the boundary of AdS_3 at $\rho = \infty$. It is now convenient to introduce new coordinates where the AdS_3 metric takes the form

$$ds_{AdS_3}^2 = -du^2 + d\chi^2 - 2 \sinh 2\sigma dud\chi + d\sigma^2 \tag{2.5}$$

These coordinates arise by viewing AdS_3 as the $SL(2, R)$ group manifold parametrized as

$$g = e^{iu\sigma_2} e^{\sigma\sigma_3} e^{\chi\sigma_1} \tag{2.6}$$

where σ_i are the usual Pauli matrices. In contrast, to get the metric in (2.4) we set

$$g = e^{i\sigma_2(\frac{\tau+\varphi}{2} - \frac{\pi}{4})} e^{\rho\sigma_3} e^{i\sigma_2(\frac{\tau-\varphi}{2} + \frac{\pi}{4})} \tag{2.7}$$

¹A CFT can have Weyl anomalies that are local and thus should not affect the results for the non-local part of the Wilson loop expectation value that we are considering.

The explicit relation between the two coordinates is

$$\begin{aligned} \sinh 2\sigma &= -\sin(\tau - \varphi) \sinh 2\rho \\ e^{4iu} &= e^{2i(\tau+\varphi)} \frac{\cos(\tau - \varphi) + i \cosh 2\rho \sin(\tau - \varphi)}{\cos(\tau - \varphi) - i \cosh 2\rho \sin(\tau - \varphi)} \\ \sinh 2\chi &= \frac{\cos(\tau - \varphi) \sinh 2\rho}{\sqrt{1 + \sin^2(\tau - \varphi) \sinh^2 2\rho}} \end{aligned} \tag{2.8}$$

One can see that in the new coordinates the Wilson loop is at $\chi \rightarrow \pm\infty$ and at $\sigma = 0$. Thus in the new coordinates (2.5), the two commuting non-compact symmetries of the problem correspond to explicit isometries. Namely, to shifts in u and χ . In particular, we have that the Hamiltonian in the new coordinates corresponds to $\Delta - S = i\partial_u$. Note that the $SL(2)_L \times SL(2)_R$ isometries of AdS_3 act on g as left and right matrix multiplication. The two commuting isometries corresponding to u and χ translations are embedded in $SL(2)_L$ and $SL(2)_R$ respectively.

Since the Wilson loop is at the boundary, we end up with a configuration where we have some color electric flux in the u, χ directions. It turns out that the flux is localized in the direction σ due to the warp factor in (2.5). Thus the flux leads to a constant energy density per unit χ and the energy is extensive in χ .

Let us now explain in more detail why the energy is confined in the direction σ . Note that for large σ the $\sinh 2\sigma$ term in (2.5) dominates. For very large and positive σ we have that $ds^2 \sim e^{2\sigma} dud\chi$. Thus we can view u and χ as lightcone coordinates of a two dimensional space with a large warp factor or gravitational potential. Thus the flux is pushed towards smaller values of σ . For very large and negative σ we can make a similar argument. The conclusion is that the flux is concentrated around $\sigma \sim 0$. Note that the direction ψ in (2.4) is compact, so that the flux cannot dissipate in this direction either. In appendix A we consider explicitly the case of a $U(1)$ gauge field and we show that indeed the flux is confined to the region around $\sigma \sim 0$. The computation in appendix A also provides a derivation for the one loop computation for the energy density and $f(\lambda)$.

The conclusion of this discussion is that the expectation value of the Wilson loop is divergent because of the infinite extent of the χ direction, but it has a finite energy density per unit distance in the χ direction, due to the fact that the flux does not dissipate due to the gravitational potential in the σ direction which leads to a confining potential. More precisely the energy, $\Delta - S$, for the configuration is

$$\Delta - S = \frac{1}{2} f(\lambda) \Delta\chi \tag{2.9}$$

where the $1/2$ is simply a convention.

Now we would like to relate $\Delta\chi$ to S . First notice that the spin is an isometry of AdS_3 in (2.4). The spin generator written in terms the coordinates in (2.5) has terms which go like $e^{\pm 2\chi}$ (see the explicit expressions in appendix A). In order to see this, note that 2χ translations are conjugate to the generator $\frac{\sigma_1}{2}$ (2.6), while the other $SL(2)_R$ generators have charges plus or minus one under the action of this generator. This implies that we have exponentials of the form $e^{\pm 2\chi}$. In the case that we have finite spin, we expect that

the quark and the anti-quark are sitting around $\pm\chi_0$ respectively. This would lead to a spin of the form $S \sim e^{2|\chi_0|}$, or

$$\Delta\chi = 2\chi_0 = \log S \tag{2.10}$$

Another way to say this is to note that the contribution to the spin from the flux alone goes as an integral over χ of a factor that grows like $e^{2\chi}$. So, if the configuration has spin S we need to cut off this integral around $e^{2|\chi_0|} \sim S$. Then for a quark-antiquark high spin operator we get

$$\Delta - S = \frac{1}{2}f(\lambda) \log S \tag{2.11}$$

while for a single trace operator made of adjoint fields we get

$$\Delta - S = f(\lambda) \log S \tag{2.12}$$

Thus we see that $f(\lambda)/2$ has the interpretation of the energy density of the flux configuration along u, χ in the coordinates (2.5).

We should also mention that one can consider an operator containing n fast moving partons. In the planar limit, the flux joins neighboring partons and we have a contribution going like $\Delta - S = n\frac{f}{2} \log S$ if the spins of all partons are equal. This type of configurations were studied in [15, 16].

2.2 Finite N and relation to two dimensional QCD

Let us make now some remarks about the finite N case. If we have dynamical quarks, or if we consider the adjoint case (2.12) then we see that we can nucleate colored particles that can screen the flux. In that case the energy no longer scales like $\log S$. Of course, this is energetically convenient only once $f(\lambda) \log S$ (or $f(\lambda)\Delta\chi$) becomes of order one. Thus, within the context of perturbation theory, where $f(\lambda)$ is very small, we can ignore this issue and argue that we have the $\log S$ scaling also for finite N .² However, for strong enough coupling the leading twist operators are not single trace operators. This is relevant for deep inelastic scattering processes in strongly coupled field theories [17]. On the other hand, in $\mathcal{N} = 4$ super Yang Mills, we can consider the lightlike Wilson loop operator for fundamental *external* quarks. In this case, the flux configuration is protected by a Z_N symmetry, the center of the gauge group. Thus in $\mathcal{N} = 4$ super Yang mills we have a well defined problem in terms of which we can define $f(\lambda, N)/2$, for arbitrary values of the arguments. We can also consider the strong coupling limit and relate this to a 't Hooft loop. We can also consider the function f for higher representations. In the large N limit the result is simply n times the result for the fundamental, where n is the number of boxes and anti-boxes for the representation. On the other hand, for finite N the result depends only on the N -ality (charge under Z_N) of the representation. Such configurations were considered at strong coupling in [18] using D5 branes in $AdS_5 \times S^5$ which are very similar to the ones appearing in the discussion of 1/2 BPS Wilson loops [19].

²If N is finite the coefficient for the adjoint operator is not equal to twice the coefficient for quark-antiquark operator.

In fact, it is interesting to consider the question of computing f for different representations at weak coupling. Once we view the problem in the coordinates we have proposed (2.4), (2.5), we see that we can do a Kaluza Klein reduction on to the directions parametrized by u and χ . The u -energies of various modes are given by the values of $\Delta - S$. Thus we see that all modes in the field theory are massive except for the gauge field along the u, χ direction. Thus we have a reduction to a 2d QCD problem. This leads to an effective low energy action

$$S = -\frac{1}{4g_2^2} \int dud\chi Tr[F^2] + \dots \tag{2.13}$$

where the dots indicate higher dimension operators we will describe below. The effective two dimensional coupling comes from integrating out all the Kaluza Klein modes. We thus get

$$\frac{1}{g_2^2} = \frac{\pi^2}{g_4^2} (1 + (const)g_4^2 N + \dots) \tag{2.14}$$

where g_4 is the four dimensional coupling. In principle this can include planar as well as non planar contributions.³

Notice that this effective field theory description is correct as long as there is large separation between the Kaluza Klein scale which is of order 1 and the 2d QCD scale which is of the order of $g_2^2 N$. This will be the case as long as we are at weak coupling. If the 2d QCD lagrangian were the full description, then we would conclude that for a general representation R we have Casimir scaling for f , since that is what we get in 2d QCD [20]

$$\frac{f}{2} = \frac{g_2^2}{2} C_2(R) \tag{2.15}$$

However, this is not the full story. As we integrate out the massive fields, we can get other operators beyond F^2 . These operators were represented as dots in (2.13). The first operators we can write down which are consistent with the symmetries of the problem are

$$S = -\frac{1}{4g_2^2} \int dud\chi Tr[F^2] + cN \int Tr[F^4] + c' \int Tr[F^2]Tr[F^2] \tag{2.16}$$

where c, c' are numerical constants. Such operators lead to a violation of Casimir scaling at four loops. Thus we have the prediction that in any theory we would get Casimir scaling up to three loops, and then at four loops we will get a violation of Casimir scaling. We also see that this effective field theory description breaks down when the 2d QCD scale gets to be of the order one, namely, when $g^2 N \sim 1$. In that case we should consider the full theory.

2.3 Conformal gauge theories in other dimensions

The argument given here for the logarithmic scaling of anomalous dimensions of high spin operators is completely geometrical and can also be generalized to other dimensions where we can have conformal field theories with a gauge symmetry. For a field theory in D dimensions

³Note that if we start with a $U(N)$ theory we have to distinguish between the g_2 for $SU(N)$ and the one for $U(1)$.

we can do a Weyl transformation of the metric to write the metric as $AdS_3 \times S^{D-3}$ and repeat the above arguments. In the case of $D = 3$ we get two copies of AdS_3 which are connected through the boundary conditions for the fields. In the two dimensional case, it might also be possible to find an argument (see [21] for a related problem) but we leave this for the future.

2.4 High spin limit of double trace operators

Notice that a crucial part of the argument leading to the logarithmic scaling is the presence of a conserved flux. In cases where we do not have a conserved flux we do not have a $\log S$ scaling. As an example, we can consider the behavior of double trace operators in a conformal theory, such as $\mathcal{N} = 4$ super Yang Mills. Here we consider operators of the schematic form $\mathcal{O}_d = \mathcal{O}_s(\mathcal{D}^{\leftrightarrow})^S \mathcal{O}_s$, where \mathcal{O}_s are gauge invariant single trace operators. To leading order in N the dimension of these operators goes like $\Delta - S = 2\Delta_s$, where Δ_s is the dimension of the single trace operator. Here we consider the $1/N^2$ correction to this result.

The discussion we had above regarding the symmetries applies to this case too. However, in this case we do not have a color flux along χ . Instead we are exchanging color neutral states between the states created by the two operators \mathcal{O}_s . Let us first study which states can be exchanged. Our first task is to determine the energies of the possible states. The u -energies are simply given by $\epsilon = \Delta - S$, where Δ is the energy of the state in the cylinder. Thus we expect to find that we get a potential of the form $V \sim e^{-(\Delta_e - S_e)\Delta\chi}$, where Δ_e, S_e are the conformal dimensions and spin of the exchanged particles.⁴ Using (2.10) we find that for large spin S we get

$$\Delta - S = 2\Delta_s + \text{const} \frac{1}{N^2} \frac{1}{S(\Delta_e - S_e)} \tag{2.17}$$

The leading power comes from the operator with the lowest value of $\Delta_e - S_e$. Notice that Δ_e and S_e are not large. In general the possible exchanged states are subject to selection rules due to the symmetries of the operator \mathcal{O}_s , thus we cannot simply take the operator with lowest $\Delta_e - S_e$ in the spectrum of the theory. However, we can always exchange the stress tensor operator, which has twist $\Delta_e - S_e = 2$. Thus, the power of S in (2.17) is bounded by $1/S^2$. Notice that this result is valid whether or not the gauge theory has an AdS dual or not. Our argument is purely field theoretic and it is valid for any conformal theory, including non-gauge theories.

The result (2.17) agrees with the more detailed analysis performed in [22, 23] which used the gravity description.⁵ The more detailed analysis of [23] makes it possible to consider more general operators, but the leading dependence on S for large spin is fixed by this argument.⁶

⁴In performing this argument we have implicitly assumed that the operator that performs χ translations has eigenvalues which are related to the ones of the operator performing u translations. This can be understood from analytic continuation in the metric (2.5).

⁵See section 4.6 in [23] and take $h \sim S, \bar{h} \sim \text{small}, \Delta \rightarrow \Delta_e, j \rightarrow S_e$.

⁶Maybe is possible to make a general field theory argument for all the operators considered in [23].

As we will see below, these results also explain why do not get double log Sudakov divergences in theories without conserved fluxes, such as the ϕ^3 theory in six dimensions considered in [7].

In summary, the reason for the $\log S$ is simply that the configuration develops an additional symmetry in the infinite spin limit. This symmetry becomes manifest when we do a Weyl rescaling of the metric. We then see that the energy is extensive along the coordinate conjugate to this symmetry generator. For finite spin, we have only a finite range for this coordinate, a range proportional to $\log S$.

2.5 Sudakov factors and the cusp anomalous dimension

Another issue that can be understood by performing Weyl transformations in the metric and coordinate redefinitions is the behavior of soft divergences in scattering amplitudes. It is well known that exclusive scattering amplitudes of colored (or charged) have infrared divergencies due to the emission of low energy gluons (or photons). These IR divergences disappear when we consider a physical observable (see [8] for example), but are sometimes replaced by explicit dependence on detector resolutions or parameters entering in the definition of jet observables. For this reason a great deal of effort was devoted to understanding the structure of these divergences. These divergences can be resummed into an expression of the form [3, 4, 7, 5, 6, 24]

$$\mathcal{A} \sim e^{-h(\lambda)(\log \mu_{\text{IR}})^2 - h'(\lambda) \log \mu_{\text{IR}}} \tag{2.18}$$

where h is some function of the coupling. In a planar gauge theory the color of each gluon is correlated with the anticolor of the next and so on. For each consecutive pair we get a factor of the form (2.18), with a function h given by $f/4$. These double logarithmic divergences can be computed by replacing the hard gluons by Wilson loops [4] (see also [25] for a more systematic discussion). In particular, we have light-like Wilson loops with a cusp. We would like to see that this cuspy Wilson line in the fundamental representation has a behavior of the form

$$\langle W \rangle \sim e^{-\frac{f(\lambda)}{4}(\log \mu_{\text{IR}}/\mu_{\text{UV}})^2} \tag{2.19}$$

where μ_{IR} and μ_{UV} are the UV and IR cutoffs. This behavior depends on how we introduce the UV cutoff. Here we are introducing it in the way that arises when we consider scattering amplitudes, where the UV cutoff appears when we can no longer replace the hard particles by Wilson lines.⁷ In other words, imagine we have two gluons coming out of an interaction region around the origin with momenta $(k_-, 0)$ and $(0, k_+)$. We can then replace the gluon with momentum k_- with a Wilson line along $x^+ = t+x > 0$ and a fuzziness in the direction x^- given by $\Delta x^- \sim 1/|k_-|$. the other gluon gives rise to a Wilson line along $x^- = t-x > 0$.

In this case we can start with the coordinates

$$ds^2 = ds^2_{AdS_3 \times S^1} = \frac{-dt^2 + dx^2 + dr^2}{r^2} + d\psi^2 = \frac{ds_{R^{1,3}}^2}{r^2} \tag{2.20}$$

⁷If we were to choose a boost invariant UV and IR regulator we would get a divergence due to the boost invariance. The UV regulator we choose here is not boost invariant.

We can now choose the coordinates

$$\begin{aligned} \frac{t \pm x}{r} &= \sin \alpha e^{\pm \gamma}, \\ \frac{1}{r} &= \cos \alpha e^{-\tau} \\ ds_{AdS_3}^2 &= -d\alpha^2 + \sin^2 \alpha d\gamma^2 + \cos^2 \alpha d\tau^2 \end{aligned} \tag{2.21}$$

In this case the Wilson loop sits at $\tau = -\infty$ and $\chi \rightarrow \pm\infty$. These coordinates cover only a portion of AdS_3 , namely the region $0 < t^2 - x^2 < r^2$. We can cover other regions by going through $\alpha \sim 0$ and $\alpha = \pi/2$ which are only coordinate singularities, but are otherwise smooth surfaces. For example, if we set $\alpha \rightarrow i\alpha'$, we go to the region $t^2 - x^2 < 0$ and with $\alpha \rightarrow i\alpha'' + \pi/2$ we go to the region $t^2 - x^2 - r^2 > 0$ which is the forward light cone of the point $t = x = r = 0$ in AdS_3 .

The region parametrized by (2.21) intersects the boundary along $t = \pm x > 0$, which coincides with the cuspy Wilson loop we want to consider. This is a time dependent background and we consider first the ordinary AdS_3 vacuum which leads to a particular state at both the future and past horizons in (2.21). Then, starting and ending with such a vacuum, we insert a Wilson line operator at $\alpha = \pi/4$, $\tau = -\infty$ and $\chi \rightarrow \pm\infty$. We then have flux lines joining these two asymptotic regions. We now note that we can get the metric in (2.21) by performing an analytic continuation of the metric in (2.5)

$$\alpha = i\sigma + \pi/4, \quad \tau = iu - \chi, \quad \gamma = iu + \chi \tag{2.22}$$

Notice that the particular point $\sigma = 0$ corresponds to $\alpha = \pi/4$. This implies that the flux configuration that appears in the Sudakov computation leads to a factor

$$\mathcal{A} = e^{-\frac{f(\lambda)}{4} \int d\tau d\gamma} \tag{2.23}$$

where f is the same as the function that appeared above (2.9). The factor of four in (2.23) is due to the change in measure $\int dud\chi \rightarrow -i \int d\tau d\gamma$. This analytic continuation is also responsible for the fact that we get a real exponent in (2.23). Thus, this analytic continuation explains the connection between the f that appears in the Sudakov factor and the function f that appears for high spin operators [4]. A similar analytic continuation was also used in [26] to argue, from the AdS side, that the same function appears on both calculations.

Now we need to relate the range of τ and γ in (2.23) to the IR and UV cutoffs. We expect the IR cutoff to be boost invariant so that we get $\tau < -\log \mu_{IR}$. If we want to relate this computation to the computation of gluon scattering amplitudes, then the UV cutoff is not boost invariant since the approximation of replacing a hard gluon by a Wilson line fails if we do a very high boost since the gluon ceases to be hard. Thus, we get a constraint of the form

$$\tau_{UV} \equiv -\log \mu_{UV} < \tau \pm \gamma \tag{2.24}$$

In the application to gluon scattering the UV cutoff is related to the momentum transfer between the two gluons generating the flux, $\mu_{UV}^2 \sim -s$.⁸ Thus we are integrating

⁸We have the cutoffs $\Delta x^\pm \sim 1/|k_\pm|$. Since the final answer will be boost invariant we can take $\Delta x^\pm \sim 1/\mu_{UV}$, with $\mu_{UV}^2 = -s$.

over a triangular shaped region. The regions near each boundary in (2.24) correspond to the collinear regions. Near these boundaries we can replace only one of the gluons by a Wilson line but not both of them. Thus, we are integrating over a domain of the form

$$\int d\tau d\gamma = \int_{\tau_{UV}}^{\tau_{IR}} d\tau \int_{-(\tau-\tau_{UV})}^{\tau-\tau_{UV}} d\gamma = \left[\log \frac{\mu_{IR}}{\mu_{UV}} \right]^2$$

$$\mathcal{A} \sim \exp \left\{ -\frac{f}{4} \left[\log \frac{\mu_{IR}}{\mu_{UV}} \right]^2 \right\},$$

$$\tau_{UV} = -\log \mu_{UV}, \quad \tau_{IR} = -\log \mu_{IR} \tag{2.25}$$

where we have neglected single log terms that can arise from the boundaries in (2.24). By scale invariance, such region are expected to contribute with a term linear in τ . Thus we get an additional term of the form [5]

$$\exp \left\{ \frac{1}{2} g(\lambda) \int_{\tau_{UV}}^{\tau_{IR}} d\tau \right\} \tag{2.26}$$

We should also note that using similar reasoning, but now considering the matching between different regions of AdS_3 in (2.20) that we discussed above, we could consider a non-lightlike cusp Wilson line with a boost angle γ . For a large angle γ , then we expect an expression of the form

$$\langle W \rangle \sim e^{-\frac{f}{4} \Delta\chi} \int d\tau \sim e^{-\frac{f}{4} \gamma \log \frac{\mu_{UV}}{\mu_{IR}}}, \quad \Delta\chi = \gamma \tag{2.27}$$

Which is the well known result [4], that the cusp anomalous dimension of a non-lightlike Wilson loop [27], is linear in γ for large γ . The reader should not be confused by the fact that one calls f a ‘‘cusp anomalous dimension’’, though the scaling with the cutoff is different for a light-like cusp (2.19) than for the original, non-lightlike, cusp discussed in [27], which has a single log.

2.6 Non-conformal theories

It is also possible to understand the above behavior in non-conformal theories. Here the new element is that the gauge theory has a scale, which we parametrize with Λ . We can think of this scale as the UV cutoff, where we set the coupling constant. The coupling can then be evolved to a lower scale via the RG equation. Then, as we do a Weyl transformations of the metric by some function $\Omega(x)$, $ds^2 = \Omega^2(x) ds'^2$ we find that the new theory in the new metric, ds'^2 , has a scale $\Lambda' = \Lambda\Omega(x)$. This means that in the theory on the new space ds'^2 we are setting the coupling to a constant at the scale Λ' which depends on the position. If we now were to choose a constant cutoff Λ'' in the new theory we would find that the coupling on the new cutoff scale Λ'' is not constant but x dependent. This dependence can be obtained by solving the RG equation to relate the constant coupling at scale $\Lambda'(x)$ to the scale Λ'' . As long as the starting Λ is sufficiently large, the value of Λ' is large enough so that the dependence on x is be very slow and we can use the ordinary RG equation for a constant coupling. In appendix B we discuss more explicitly the simpler case where we map a non-conformal theory between the plane, R^4 , and the cylinder $R \times S^3$.

In our case, in (2.20) we find that $\Omega \sim r$ and thus the new scale is

$$\Lambda' = \Lambda r = \Lambda \frac{e^\tau}{\cos \alpha} \quad (2.28)$$

The α dependence is not too important for us. The τ dependence implies that the Hamiltonian generating τ translations is τ dependent and the value of the Wilson loop involves an integral over τ . When we do that integral, it is important to remember that the range of γ also depends on τ and the UV and IR cutoffs. Thus, when we repeat the above computations we find that g and f are the τ dependent eigenvalues of the τ -Hamiltonian. They depend on the coupling, which itself depends on τ through the β function equation. If we use dimensional regularization in order to cutoff the IR divergencies, then we should use the β function in $4 + \epsilon$ dimensions to determine the τ dependence of the coupling constant. The bottom line is that we obtain an expression similar to the above one but the function $f(\lambda(\tau))$ depend explicitly on the “time” τ and should be placed inside the integrals. These expressions agree with the expressions in [5, 24].

If we concentrate on the leading dependence on the IR cutoff, then we are integrating in τ up to $\tau = -\log \mu_{\text{IR}}$ and for each value of τ the range of γ is $\Delta\gamma \sim 2(\log \mu_{\text{UV}} - \tau)$. Thus we have

$$\langle W \rangle \sim e^{-\int_{-\log \mu_{\text{UV}}}^{-\log \mu_{\text{IR}}} d\tau \frac{f(\lambda(\tau))}{4} \Delta\chi(\tau)} \sim e^{-\int_{-\log \mu_{\text{UV}}}^{-\log \mu_{\text{IR}}} d\tau \frac{f(\lambda(\tau))}{4} 2(\log \mu_{\text{UV}} - \tau)} \quad (2.29)$$

This agrees with the general expressions derived in [5, 24], for the leading IR divergence of gluon scattering amplitudes.

Using the *AdS/CFT* one can consider the computation of Wilson loops with cusps in non-conformal theories. This was done for the Klebanov-Strassler cascading theory in [28] and in $4 + \epsilon$ dimensions in [29].

2.7 High spin operators at strong coupling

We now consider $\mathcal{N} = 4$ super Yang Mills at strong coupling and analyze it using the gravitational dual. From our general discussion we concluded that $f(\lambda)$ can be computed in terms of a light-like Wilson loop. It is convenient to slice AdS_5 in coordinates where the boundary is manifestly $AdS_3 \times S^1$

$$\begin{aligned} ds^2 &= \cosh^2 \zeta ds_{AdS_3}^2 + \sinh^2 \zeta d\psi^2 + d\zeta^2 \\ &= \cosh^2 \zeta [-du^2 + d\chi^2 - 2 \sinh 2\sigma dud\chi + d\sigma^2] + \sinh^2 \zeta d\psi^2 + d\zeta^2 \end{aligned} \quad (2.30)$$

These coordinates cover all of AdS_5 and the boundary sits at $\zeta \rightarrow \infty$.

In the gauge theory we considered a configuration with flux in the $u\chi$ direction (2.5). This gives rise to a string extended along the $u\chi$ directions of AdS_3 . The warp factor in the ζ direction pushes the string to $\zeta = 0$, which is a $U(1)_\psi$ symmetric point. In addition the warp factor in the σ direction pushes the string to $\sigma = 0$.

There are some interesting features of this string. First, its tension gives us the energy density. Thus, simply the tension of the string gives us the strong coupling behavior for the cusp [14, 30]

$$\frac{f}{2} = \frac{T}{2\pi\alpha'} = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi} \quad (2.31)$$

We see that we get the result in a direct way without solving any equations.

Second, we can easily consider small fluctuations around this string configuration [11]. We can see that for quadratic fluctuations we have a boost symmetry in the u, χ directions. This is not a symmetry of the full problem, but it is a symmetry of the theory at the quadratic level. We can easily find the bosonic excitations and we can compute their masses. We find that there are five massless goldstone bosons associated to the motion on S^5 . The oscillations in the σ direction are described by a massive goldstone field with mass $m^2 = 4$ that comes from the $SL(2)_L$ symmetries that the string breaks.⁹ In other words, the creation and annihilation operators for the modes of the σ field on the worldsheet come from J_L^\pm in $SL(2)_L$, recall that $2J_L^3 = i\partial_u$ is the energy. This corresponds to oscillations in the σ direction inside AdS_3 . Then there are two bosons of $m^2 = 2$ associated to motion in $\zeta e^{i\psi}$, these are not obviously Goldstone bosons. Nevertheless one can view them as Goldstone bosons according the following heuristic argument. The full theory has conformal symmetries which are not isometries of $AdS_3 \times S^1$. In particular we have conformal generators in the spin $(\frac{1}{2}, \frac{1}{2}, \pm 1)$ under $SL(2)_L \times SL(2)_R \times U(1)_{S^1}$. These symmetries are broken by the string. They create modes with wavefunctions of the form $e^{-iu \pm \chi}$. We see that these are exponentially growing in the χ direction and thus would carry momentum $p = \pm i$. Thus, the dispersion relation should be such that $\epsilon(p = \pm i) = 1$. If the dispersion relation is relativistic, then we get $m^2 = 2$. This argument is heuristic because we are talking about non-normalizable modes. Now let us turn to the fermions. All the fermions have the same mass since they have to transform under the spinor representation of $SO(6)$ and the lowest dimensional representations have four complex dimensions, corresponding to eight real fermions. They all have $m = 1$ which can be viewed again as goldstone fermions. Their mass is fixed by the transformation properties of the supercharges under $SL(2)_L$, where $SL(2)_L$ are the left isometries of AdS_3 . The advantage of viewing them as Goldstone bosons or fermions is that their energies stay with these values (at zero momentum) as long as the symmetry is not restored.¹⁰ From this spectrum of masses it is straightforward to compute the vacuum energy and we obtain the one loop contribution [11]

$$\frac{f_1}{2} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{1}{2} \left[5|p| + \sqrt{p^2 + 4} + 2\sqrt{p^2 + 2} - 8\sqrt{p^2 + 1} \right] = -\frac{3 \log 2}{2\pi} \quad (2.32)$$

One can also check explicitly that this string configuration breaks all the supersymmetries. Thus it is different from the string configuration corresponding to the circular Wilson loop which is BPS.

3. The $O(6)$ sigma model from string theory

In this section we consider further the worldsheet theory describing the string associated with highly spinning operators or lightlike Wilson loops. We consider a string stretched

⁹Goldstone bosons can be massive when the broken symmetry does not commute with the Hamiltonian.

¹⁰We will see that the 5 massless modes will actually get a mass non-perturbatively in the α' expansion and the $SO(6)$ symmetry will be restored.

along the u, χ coordinates in (2.30), and we work in static gauge. Notice that the theory is not invariant under boosts in the χ and u directions, but it does become boost invariant at low energies. In fact, the spectrum we discussed above is precisely boost invariant. Let us now imagine taking a low energy limit where we look at the system at distances (in χ) much larger than one, which is the mass of the fermions and the order of magnitude of the mass of the massive bosons. In this case only the massless excitations survive. In two dimensions, massless fields have large fluctuations which lead to interesting dynamics in the IR. In our case, the massless fields describe an S^5 . In other words, they describe the $O(6)$ sigma model. This is a model where the coupling becomes strong in the IR and the theory develops a mass gap. Moreover, this is an exactly solvable theory [31]. The scale set by the mass of the fermions (and massive bosons) acts as a UV cutoff for the $O(6)$ sigma model, where the $O(6)$ theory merges into the full $AdS_5 \times S^5$ sigma model.

When we compute the cusp anomalous dimension at strong coupling we are computing the vacuum energy of this theory. Of course, the vacuum energy in the $O(6)$ sigma model is UV divergent. In our case, this UV divergence is cut off at the scale where the fermions start contributing. We can see this explicitly in the one loop result (2.32). Thus the vacuum energy does not seem to have a clear contribution that comes purely from the $O(6)$ sigma model. A two loop computation was attempted in [32].

There are some interesting features of this relation to the $O(6)$ model. First, note that the massive excitations of the theory transform in the vector representation of $O(6)$, thus it is natural to identify them with the fundamental scalars ϕ^I of the gauge theory. It looks like such excitations should appear naturally as we compute the spectrum of charged operators around the lowest twist high spin operator.

This relation to the $O(6)$ sigma model allows us to perform all orders computations in AdS by focusing on the right observable. We choose an observable which receives most of its contribution from the low energy region described by the $O(6)$ sigma model. There is an interesting concrete set of operators that has been studied recently [11, 12] which has a limit that can be explored in terms of the $O(6)$ theory. These are operators which carry large spin S and also large charge J , where J scales like $\log S$. In other words, consider single trace operators in planar $\mathcal{N} = 4$ super Yang Mills with

$$S, J \rightarrow \infty, \quad j \equiv \frac{J}{2 \log S} = \text{fixed} \tag{3.1}$$

(the factor of 2 is for convenience). Using the arguments in the previous section, which connect the spin S to the extension of the string (or corresponding field theory configuration) in the χ direction we see that in this limit we have a configuration with finite current density along the χ direction on the worldsheet. We then conclude that the anomalous dimensions scale as $\log S$ and that

$$\lim_{S \rightarrow \infty} \frac{\Delta - S}{\log S} = f(\lambda) + 2\epsilon(\lambda, j) \tag{3.2}$$

where $f(\lambda)$ is the cusp anomalous dimension and $2\epsilon(\lambda, j)$ is the additional energy due to the $SO(6)$ charge density j . Note that $\epsilon(j = 0) = 0$. The argument of the previous section implies that this is the right scaling for all values of λ and j .

Note that at strong coupling we can consider the same string we discussed above, which is stretched along the u and χ directions. The string carries a current density proportional to j since $2 \log S = \Delta\chi$ is the length of the folded string corresponding to a single trace operator of spin S . Similarly the factor of 2 in (3.2) is chosen so that $\epsilon(j)$ is the energy density along a single string stretched along the χ direction.

We can compute $\epsilon(j)$ using the $O(6)$ sigma model in the regime where the characteristic time variations of the angular coordinates are much smaller than the mass of the fermions. We have

$$j = \frac{\sqrt{\lambda}}{2\pi} \dot{\varphi} \tag{3.3}$$

We require $\dot{\varphi} \ll 1$, thus we want $j \ll \sqrt{\lambda}$ for starting to trust the $O(6)$ results. The classical sigma model result is

$$\epsilon(j) = \frac{\sqrt{\lambda}}{4\pi} \dot{\varphi}^2 = \frac{\pi}{\sqrt{\lambda}} j^2 \tag{3.4}$$

The strong infrared dynamics of the $O(6)$ sigma model generates a mass gap

$$m = k\lambda^{1/8} e^{-\frac{1}{4}\sqrt{\lambda}} \left[1 + o\left(\frac{1}{\sqrt{\lambda}}\right) \right] \tag{3.5}$$

where k is a constant that depends on the details of how the $O(6)$ model is embedded in the full $AdS_5 \times S^5$ string sigma model. This formula is valid at large $\sqrt{\lambda} \gg 1$ so that the $O(6)$ sigma model can be suitably decoupled from the rest.

Let us specify more precisely the decoupling limit that gives the $O(6)$ model. We take the limit $S \rightarrow \infty$ with j fixed (3.2). We then take the limit

$$\lambda \rightarrow \infty, \quad j \rightarrow 0, \quad \text{with} \quad \frac{j}{m} = jk^{-1}\lambda^{-1/8}e^{\frac{1}{4}\sqrt{\lambda}} = \text{fixed} \tag{3.6}$$

In this limit we find that

$$\epsilon(j) = j^2 \mathcal{E}(j/m) \tag{3.7}$$

where we used dimensional analysis. The function \mathcal{E} can be determined purely in terms of the $O(6)$ sigma model [13]. For large j/m this function can also be expanded using $O(6)$ perturbation theory. Thus, we can use the $O(6)$ results to compute the α' expansion of this observable.

The problem of computing the energy of a configuration with constant current density was considered in [13]. These authors derived an integral equation determining the energy as a function of the chemical potential h for the charge j . They found that

$$F(h) = h^2 \mathcal{F}(h/m) = -\frac{h^2}{2} \left(-\beta_1 \log(h/m) + \frac{\beta_2}{\beta_1} \log \log(h/m) + c + \tilde{c} \frac{\log \log(h/m)}{\log(h/m)} + \dots \right) \tag{3.8}$$

where the coefficients β_1 and β_2 can be related to the one and two loop beta function coefficients in the $O(6)$ theory. Of course the first two terms can also be easily computed using perturbation theory in the sigma model. But the computation of c amounts to a

computation of the mass gap which is more involved [13]. In appendix C we review [13] and give the values of these coefficients for an $O(N)$ model. In appendix C we also show that the structure of the perturbative series can be used to fix the coefficients of the logarithmic terms (for example \tilde{c}). The energy density discussed above can be computed by first computing j from (3.8) and then performing the Legendre transform to get $\epsilon(j) = F(h) + jh$. In this fashion we can obtain the function \mathcal{E} in (3.7).

This gives a precise prediction that could be used to test the BES/BHL [33, 9] guess for the S-matrix to all orders in the α' expansion, and probably in an exact way. Hopefully, all that one needs to perform this comparison is to have a closer look at the Bethe equations in the $SL(2)$ subsector and repeat the steps in [34, 9, 35] keeping $J/\log S$ constant.

3.1 $O(6)$ free energy and comparison to string theory

One loop string theory computations on a related regime have been considered by Frolov, Tirziu and Tseytlin in [11]. They considered closed string configurations with

$$y \equiv \frac{2\pi}{\sqrt{\lambda}} j \tag{3.9}$$

fixed.¹¹ As a check of what we have been saying we show that the limit of small j , y of the formulas in [11], match precisely the expectation from the $O(6)$ side. In addition, using their computation we can fix the coefficient k in (3.5), which is sensitive to threshold corrections.

In the small j limit their tree level and one loop results read

$$\epsilon(j) = \frac{\Delta - S - f(\lambda) \log S}{2 \log S} = y^2 \left(\frac{\sqrt{\lambda}}{4\pi} - \frac{1}{\pi} \log y + \frac{3}{4\pi} \right) \tag{3.10}$$

The tree level result is simply (3.4).

For large values of j , $j/m \gg 1$ (but still with $j \ll \sqrt{\lambda}$, so that the $O(6)$ description is valid), and using the relations (3.5), (3.8) and (3.9), we can expand the energy density in powers of $\frac{1}{\sqrt{\lambda}}$. We find

$$\begin{aligned} \epsilon(\lambda, y)_{0+1} &= y^2 \left[\frac{\sqrt{\lambda}}{4\pi} + \frac{\beta_1}{2} \log(y) + k_1 \right], \\ 2k_1 &= -c - \beta_1 \log k + \frac{\beta_2}{\beta_1} \log(-2\pi\beta_1) \end{aligned} \tag{3.11}$$

Let us focus first on the $\log y$ term whose coefficient depends on β_1 . We see that for the correct $O(6)$ value, $\beta_1 = -2/\pi$, we obtain agreement with (3.10) [11]. The constant piece, k_1 , depends on k in (3.5). This is a quantity that we cannot determine by purely $O(6)$ sigma model computations, since it depends on threshold corrections that involve the other massive modes. By matching the constant piece in the one loop answer in the full theory, computed in [11] (and reproduced in (3.10)), we can actually determine

$$k = \frac{2^{1/4}}{\Gamma(5/4)} \tag{3.12}$$

¹¹They wrote their results in terms of x , with $x = 1/y$.

Note that the computation in [11] was done keeping y fixed and arbitrary, while here we are just focusing on the region $y \ll 1$ where the results can be computed using the $O(6)$ theory. Indeed, the coefficient of the $\log y$ was determined by $O(6)$ but not the constant part.

3.2 Higher loop predictions

Here we consider higher loop predictions for the energy density. At higher loops and very small y , the most important terms in the expansion are the logarithmic terms. Such terms are determined by the renormalization group equations in terms of lower order terms. The expression (3.8) is the expansion of the free energy for the first two orders in perturbation theory. This expression also determines the β function at one and two loops. We thus expect that using the renormalization group we could determine the two leading logs at each order in perturbation theory. Of course, we could just blindly expand the expression (3.8) and directly see that the two leading logs are determined. Of course, these logs are renormalizing the coupling from the UV scale where it is given in terms of $\sqrt{\lambda}$ down to the scale set by y . However, since the coupling constant expansion in the full sigma model is $1/\sqrt{\lambda}$ we might still wish to do an expansion in powers of $1/\sqrt{\lambda}$. In that case we can give the expression for the first two leading log corrections at each order (after we used (3.12)), ($\epsilon = \sum_{k=0}^{\infty} \epsilon_k$)

$$\begin{aligned} \epsilon(y)_2 &= y^2 \frac{1}{\sqrt{\lambda}\pi} (4 \log^2 y - 3 \log y + \dots) \\ \epsilon(y)_3 &= y^2 \frac{1}{\pi\lambda} (-16 \log^3 y + 6 \log^2 y + \dots) \\ \epsilon(y)_{l+1} &= y^2 \frac{1}{\pi\lambda^{l/2}} \left((-1)^{l+1} 4^l \log^{l+1} y + (-1)^l 4^{l-1} 3[(l+1)(1-h(l)) + l] \log^l y + \dots \right) \end{aligned} \quad (3.13)$$

where $h(l) = \sum_{n=1}^l 1/n$ is a harmonic sum.

It would be nice to see whether these terms, which are easily computed, contain any information about the higher order corrections for the dressing phase [36, 33, 9] when one computes this energy using a suitable generalization of the techniques in [34, 9, 35]. If that is the case, then one could test the higher order terms in the dressing phase. If the leading logs do not depend on the higher order corrections to the phase, then one would be forced to consider higher order corrections in the $O(6)$ sigma model, which are still much easier to compute than higher order corrections in the full $AdS_5 \times S^5$ string sigma model.

We should note that at each order in the $1/\sqrt{\lambda}$ expansion there are also terms which arise from higher order threshold corrections in (3.5). Such terms are not calculable in the purely $O(6)$ theory. These higher order corrections disappear if we take the decoupling limit (3.6). Only the constant k in (3.6) appears, and we have already fixed it in (3.12) using the results in [11].

3.3 Very small j limit

Note that our discussion makes sense even non-perturbatively in the $O(6)$ coupling. Thus, we can consider extremely small values of j . In this regime we have a very low charge

density, so we have well separated massive particles in the $O(6)$ theory thus make the simple prediction that

$$\epsilon(j) \sim mj, \quad j \ll m \tag{3.14}$$

where m is given in (3.5). This is valid for $\lambda \gg 1$. Thus we have particles that transform in the vector representation of $O(6)$. This is reminiscent of what happens at weak coupling, $\lambda \ll 1$, where for low j we also have an answer linear in j , $\epsilon(j) \sim j$. This is simply the statement that the scalar fields ϕ^I , which carry the $SO(6)$ charge contribute one unit to the twist $(\Delta - S)$. We see that, for very small j , the functional dependence on j is the same at weak and strong coupling. However, the coefficient is very different. The small value of the coefficient at strong coupling signals the existence of the region where the physics is described by the $O(6)$ sigma model, since there is large difference between the mass gap of the $O(6)$ particles and the mass of the fermions, which sets the scale where the $O(6)$ theory breaks down. At weak coupling, $\lambda \ll 1$, there is no such large separation and it would be wrong to use $O(6)$ formulas to compute the energy.

It would be interesting to see if one can get similar reductions to an $O(4)$ or $O(3)$ sigma models by considering strings in $AdS_3 \times S^3$ or $AdS_3 \times S^2$.

4. Conclusions

In this article we have presented a simple picture for the cusp anomalous dimension. This is a quantity that appears in various computations in gauge theories. We found it convenient to perform a Weyl transformation of the metric from $R^{1,3}$ to $AdS_3 \times S^1$, which simplifies the action of the symmetries that determine the form of the results. The cusp anomalous dimension becomes the energy density of a certain flux configuration of the gauge theory on $AdS_3 \times S^1$. It is a flux configuration that is invariant under two non-compact translation symmetries. These symmetries explain the logarithmic behavior of certain quantities. For example, the logarithmic behavior of the dimension of high spin operators, $\Delta - S = f(\lambda) \log S$, arises when the configuration has a finite extent along the coordinate conjugate to one of the translation symmetries. Similarly, the flux configuration associated to the Sudakov factor arises after performing an analytic continuation and the double logarithmic behavior arises from imposing a finite range for both of the coordinates conjugate to non-compact translation symmetries. We have also discussed how to obtain the spin dependence of anomalous dimensions of double trace operators and we found that it is given by a power determined by the twist of the lowest twist operator that couples to each single trace operator (2.17). We also explained how the weak coupling computation of the cusp anomalous dimension can be reduced to an effective two dimensional QCD problem and argued that the cusp anomalous dimension for arbitrary representations displays Casimir scaling up to three loops. The arguments are made on the field theory side and are valid for any conformal gauge theory regardless of the value of the coupling. We have discussed also the extension to the non-conformal case for the case of the Sudakov factor.

We then considered operators with high spin and charge in $\mathcal{N} = 4$ super Yang Mills and argued that in the limit where $J/\log S$ is finite we get $\Delta - S = [f + \epsilon(J/\log S)] \log S$ [11]. We showed that when $J/\log S$ is suitably small the computation of the function $\epsilon(J/\log S)$

reduces to a computation in the bosonic $O(6)$ sigma model. This relation gives us a way of obtaining exact results for the worldsheet string theory. These can, hopefully, be used to test the BHL/BES [33, 9] prediction for the phase of the S-matrix at higher loops.

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A. One loop computation for the cusp anomalous dimension

In this appendix we outline the one loop computation of the cusp anomalous dimension using the coordinates we introduced above. Of course, this is a well known computation that has been done in many ways. We perform the computation by considering a Wilson loop in the coordinates (2.5). We consider first a $U(1)$ theory, and start with a configuration with an electric flux which is non-zero only in the u, χ directions, $F_{u\chi}$, which is constant due to the Bianchi identity. We can write the action as

$$S_{\text{act}} = -\frac{1}{4g^2} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} = \frac{|F_{u\chi}|^2}{2g^2} \int \frac{d\psi du d\chi d\sigma}{\cosh 2\sigma} = \frac{\pi^2}{2g^2} \int du d\chi |F_{u\chi}|^2 \quad (\text{A.1})$$

where we have integrated over the circle parametrized by ψ in (2.20). The total energy density is

$$\frac{f}{2} = \frac{\pi^2 |F_{u\chi}|^2}{2g^2} \quad (\text{A.2})$$

The quantization condition for F can be obtained in the standard way after we say that fundamental charges couple as $e^{i\int A}$ and compute the amount of flux that this charge generates. This gives $F_{u\chi} = \frac{g^2}{\pi^2}$. We then obtain

$$\frac{f}{2} = \frac{g^2}{2\pi^2} \quad (\text{A.3})$$

We can now consider the generalization to a $U(N)$ theory. In that case we get a similar result except that we get an additional factor of N in (A.3) and there is an additional factor of two that comes from the conventional definition of the coupling. Alternatively, we can reduce the four dimensional theory to two dimensions along ψ and σ . This leads to a two dimensional theory with a coupling $g_2^2 = g_4^2/\pi^2$. Then we can use the two dimensional QCD result (2.15) with $C_2 = N/2$ for the fundamental representation so that we get the one loop value $f/2 = \frac{g^2 N}{4\pi^2}$.

A.1 Spin generators in the new coordinates

As we see in the metric (2.4) the generator that measures spin, $-i\partial_\varphi$ corresponds to an isometry in AdS_3 . We can now compute the form of this Killing vector in the other AdS_3

coordinates (2.5)

$$\begin{aligned}\Delta &= \frac{1}{2} \left(1 + \frac{\cosh 2\chi}{\cosh 2\sigma} \right) i\partial_u + \frac{1}{2} \cosh 2\chi \tanh 2\sigma i\partial_\chi + \cosh \chi \sinh \chi i\partial_\sigma \\ S &= \frac{1}{2} \left(-1 + \frac{\cosh 2\chi}{\cosh 2\sigma} \right) i\partial_u + \frac{1}{2} \cosh 2\chi \tanh 2\sigma i\partial_\chi + \cosh \chi \sinh \chi i\partial_\sigma\end{aligned}\quad (\text{A.4})$$

where we also indicated the form for Δ . We see that $\Delta - S = i\partial_u$. We see from these expressions that if we have a configuration which goes up to some distance χ_0 , then its spin scales as $S \sim e^{2|\chi_0|}$. We can understand this better if we compute the contribution to the spin of the constant flux configuration discussed above. Given a general Killing vector of the form $V = i\xi^\mu \partial_\mu$ the conserved current associated to it is given by contracting ξ^μ with the stress tensor. Using the expression for the stress tensor for a gauge field we find, after integrating over ψ in (2.5) ,

$$\begin{aligned}\Delta &= \frac{\pi}{g^2} \int d\sigma d\chi \frac{F_{u\chi}^2}{\cosh 2\sigma} \left(\frac{1}{2} + \frac{\cosh 2\chi}{2 \cosh 2\sigma} \right) \\ S &= \frac{\pi}{g^2} \int d\sigma d\chi \frac{F_{u\chi}^2}{\cosh 2\sigma} \left(-\frac{1}{2} + \frac{\cosh 2\chi}{2 \cosh 2\sigma} \right)\end{aligned}\quad (\text{A.5})$$

Of course, this flux configuration is not a good description of the quark-antiquark configuration near $\chi \sim \chi_0$. However, we argue that the dynamical particles that we put at $\chi \sim \chi_0$ still have a spin of order $S \sim e^{2|\chi_0|}$. In this fashion we connect the range of χ to the spin, via $\Delta\chi = \log S$.

B. Renormalization group and evolution on the cylinder

A conformal field theory on the plane is equivalent to a conformal field theory on the cylinder and the spectrum of anomalous dimensions of operators on the plane corresponds to the energy spectrum for the theory on the cylinder. Now, suppose that we have a non-conformal field theory on the plane. For simplicity, imagine we have a theory with a single coupling that runs $g^2(\mu)$. Then we would like to understand what type of theory we get on the cylinder.

For simplicity, consider the Euclidean theory. Then the plane and the cylinder are related by the following Weyl transformation

$$ds_{R^4}^2 = r^2 (d\tau^2 + d\Omega_3^2) = r^2 ds_{R \times S^3}^2, \quad r = e^\tau \quad (\text{B.1})$$

Thus the two metrics are related by $ds^2 = \Omega^2 ds'^2$, $\Omega = r = e^\tau$. Let us imagine regularizing the field theory on the plane with a cutoff Λ so that the value of the coupling at the cutoff is constant. On the cylinder this leads to a field theory where the cutoff $\Lambda' = \Omega(x)\Lambda$ depends on position. In our case, this leads to a cutoff $\Lambda' = e^\tau \Lambda$ which depends on the Euclidean time direction along the cylinder. The coupling constant on the cylinder is a constant at scale Λ' . This can be related to the more conventional way of defining the theory on the cylinder which uses a fixed (time independent) cutoff Λ'' . We can obtain the value of the

coupling at Λ'' by using the renormalization group equation to evolve the coupling from the scale $\Lambda'(\tau)$ to the scale Λ'' . We can apply the ordinary flat space renormalization group equation as long as the coupling is varying slowly at the scale of the cutoff. This condition reads $\partial_\tau \Lambda'/\Lambda' \ll \Lambda'$. In our case this requires that $e^\tau \Lambda \gg 1$. This says that the cutoff Λ' should be bigger than the inverse radius of the S^3 . Thus, if we want to explore the theory at $\tau \rightarrow -\infty$ (or $r \rightarrow 0$), we need to take a smaller and smaller cutoff Λ , which is of course what we expect.

In conclusion, we have a “time” dependent theory on the cylinder, where the coupling constant depends on τ . The time dependence of the coupling constant can be computed exactly if we know the exact β function. Since we have a time dependent theory on the cylinder we have a time dependent Hamiltonian. We can nevertheless diagonalize this Hamiltonian at each time and this leads to the scale dependent anomalous dimensions we have in a non-conformal theory $\Delta(g^2(\tau))$.

For theories that have a gravity dual, it is useful to understand this also from the gravity perspective. Let us start with a five dimensional metric and scalar field

$$ds^2 = w(z)^2 \frac{dz^2 + dx^2}{z^2}, \quad \phi(z) \tag{B.2}$$

We can now use the usual change of coordinate that takes us between the plane and the cylinder in the AdS case, which is

$$\frac{1}{z} = e^{-\tau} \cosh \rho, \quad \frac{r}{z} = \sinh \rho \tag{B.3}$$

so that we end up with the metric and scalar fields

$$ds^2 = w \left(\frac{e^\tau}{\cosh \rho} \right)^2 [\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2], \quad \phi \left(\frac{e^\tau}{\cosh \rho} \right) \tag{B.4}$$

We now see that in the original metric (B.2) it is natural to impose a cutoff at $z = 1/\Lambda$. This cutoff would then correspond to $e^{-\tau} \cosh \rho_c(\tau) = \Lambda$, where $\rho_c(\tau)$ is determined by this equation. From the point of view of the new theory, we would say that the cutoff is at $\Lambda' \sim e^{\rho_c(\tau)}/2 \sim \Lambda e^\tau$ if ρ_c is large. Thus we get $\Lambda' = e^\tau \Lambda$ as we had in the general discussion. Notice that the condition that the time variation was slow, translates into the condition that $\rho_c \gg 1$. If we fix the cutoff at a time independent value $\Lambda'' = e^{\rho_c''}/2$ we get that the scalar field has a time dependent value given by (B.4). In fact, we get $\phi_c \sim \phi(2e^{\tau-\rho_c''})$.

C. O(N) sigma model

In this appendix we recall some results for the $O(N)$ non linear sigma model. We consider the $O(N)$ sigma model in the presence of a chemical potential h coupled to one of the conserved charges (an $SO(2) \subset SO(6)$) and we compute the free energy $f(h) = \min_j [\epsilon(j) - jh]$. By dimensional analysis $f = h^2 \mathcal{F}(h/m)$, where m is the mass gap.

Given the two particle S -matrix for the $O(N)$ σ -model

$$S(\theta) = - \frac{\Gamma(1+x)\Gamma(1/2-x)\Gamma(1/2+\Delta+x)\Gamma(\Delta-x)}{\Gamma(1-x)\Gamma(1/2+x)\Gamma(1/2+\Delta-x)\Gamma(\Delta+x)}$$

$$x = \frac{i\theta}{2\pi} \quad \Delta = \frac{1}{N-2} \tag{C.1}$$

The thermodynamic Bethe ansatz leads to an integral equation for the free energy [13]. More precisely

$$f(h) = -\frac{m}{2\pi} \int_{-B}^B \cosh(\theta) \rho(\theta) d\theta \quad (\text{C.2})$$

where $\rho(\theta)$ satisfies the following integral equation with the following boundary condition

$$\rho(\theta) - \int_{-B}^B K(\theta - \theta') \rho(\theta') d\theta' = h - m \cosh(\theta), \quad \rho(\pm B) = 0 \quad (\text{C.3})$$

where $K(\theta) = \frac{1}{2\pi i} \frac{d}{d\theta} \log S(\theta)$.

In order to make a comparison with computations from the string sigma model, one needs to consider the regime $h/m \gg 1$ which corresponds to the weakly coupled region of the theory. This was considered in [13]. Their analysis leads to the following result for the free energy

$$\begin{aligned} f(h) &= -\frac{h^2}{2} \left(-\beta_1 \log(h/m) + \frac{\beta_2}{\beta_1} \log \log(h/m) + c + \tilde{c} \frac{\log \log(h/m)}{\log(h/m)} + \dots \right) \\ \beta_1 &= -\frac{N-2}{2\pi}, \quad \beta_2 = -\frac{N-2}{4\pi^2}, \\ c &= \frac{N-2}{2\pi} \log \left[\left(\frac{8}{e} \right)^{\frac{1}{N-2}} \frac{e^{-1/2}}{\Gamma(1 + \frac{1}{N-2})} \right] \end{aligned} \quad (\text{C.4})$$

where the notation is chosen to highlight the fact that $\beta_{1,2}$ turn out to be the one and two loop beta functions for the $O(N)$ sigma model. Next, we perform a Legendre transform and express $\epsilon = f(h) + jh = j^2 \mathcal{E}(j/m)$ in terms of the charge density $j \equiv -f'(h)$. Starting from (C.4) it is straightforward to iteratively solve for $\mathcal{E}(j/m)$

$$\begin{aligned} \mathcal{E}(j/m) &= \frac{1}{-2\beta_1 \log j/m} + \frac{1}{-2\beta_1^3 \log^2 j/m} \left((\beta_1^2 + \beta_2) \log \log j/m + \beta_1 c + \beta_1^2 \log(-\beta_1) \right) \\ &+ \frac{1}{-8\beta_1^5 \log^3 j/m} \left[4(\beta_1^2 + \beta_2)^2 (\log \log j/m)^2 - k' \log \log j/m + \text{const} \right] \\ &+ \mathcal{O} \left(\frac{1}{\log^4 j/m} \right) k' \\ &\equiv 4\beta_1 \left(\beta_1^3 + 2\beta_1 \beta_2 - 2\beta_2 c - \beta_1^2 (2c + \tilde{c}) - 2\beta_1 (\beta_1^2 + \beta_2) \log(-\beta_1) \right) \end{aligned} \quad (\text{C.5})$$

The constant piece in front of $\frac{1}{\log^3 j/m}$ depends on a higher order term, of the form $1/\log(h/m)$, in (C.4) which we have not computed. In order to make a comparison with the results of [11] we need to express our expansion parameters j and m in terms of y and the coupling constant λ using (3.9), (3.5), or

$$m = k\lambda^{-\frac{\beta_2}{2\beta_1^2}} e^{\frac{\sqrt{\lambda}}{2\pi\beta_1}} \quad (\text{C.6})$$

Expanding $\epsilon(\lambda, x)$ as a power series on λ we obtain

$$\begin{aligned} \epsilon(\lambda, x) &= y^2 \left[\frac{\sqrt{\lambda}}{4\pi} + \frac{\beta_1}{2} \log(y) + k_1 \right. \\ &\left. + \frac{\pi}{4\sqrt{\lambda}\beta_1^2} \left(4\beta_1^4 \log^2(y/(2\pi k)) + k_2 \log y + (2\beta_2^2 + 2\beta_1^3 \tilde{c}) \log(\lambda) + \text{const} \right) + \dots \right] \end{aligned} \quad (\text{C.7})$$

Where the terms k_1 and k_2 entering at one and two loops are

$$\begin{aligned}
 2k_1 &= -c - \beta_1 \log k + \frac{\beta_2}{\beta_1} \log(-2\pi\beta_1) \\
 k_2 &= 4\beta_1^2 [\beta_1^2 + \beta_2 - 2\beta_1 c + 2\beta_1^2 \log(2\pi) + 2\beta_2 \log(-2\pi\beta_1)]
 \end{aligned}
 \tag{C.8}$$

Now let us discuss some properties of the function \mathcal{F} in (C.4). This function has a structure of the form

$$\mathcal{F}(t) = f/h^2 = -\frac{1}{2} \left(-\beta_1 t + \frac{\beta_2}{\beta_1} \log t + c + \sum_{n=1}^{\infty} \sum_{m=0}^n a_{nm} \frac{(\log t)^m}{t^n} \right)
 \tag{C.9}$$

This structure is dictated by the structure of perturbation theory. Moreover, all the coefficients for terms involving logarithms, a_{nm} with $m > 0$, are determined in terms of lower order coefficients, $a_{n'm}$ with $n' < n$. In particular, the constant \tilde{c} is determined by the structure of perturbation theory

$$\tilde{c} = -\frac{\beta_2^2}{\beta_1^3}
 \tag{C.10}$$

Notice that with this value of \tilde{c} the term involving a $\log \lambda$ in (C.7) disappears. In general, the logarithmic terms in (C.9) are fixed by demanding that we can express the answer in terms of a power series expansion in terms of the effective coupling $\bar{g}^2(\mu)$ of the sigma model, with the additional condition that the dependence of the coupling constant $\bar{g}(\mu)$ on the scale μ is described by the Callan-Symanzik equation

$$\mu \partial_\mu \bar{g}^2(\mu) = \beta_1 \bar{g}^4(\mu) + \beta_2 \bar{g}^6(\mu) + \dots = \beta(\bar{g}^2(\mu))
 \tag{C.11}$$

where $\beta(g^2)$ also has a power series expansion in \bar{g}^2 . This beta function equation can be solved as

$$\frac{1}{\bar{g}^2} = -\beta_1 \log(\mu/\Lambda) + \frac{\beta_2}{\beta_1} \log(\log \mu/\Lambda) + \dots
 \tag{C.12}$$

where Λ is the dynamical scale of the theory. If we take the renormalization scale at $\mu = h$, then we can think of $\log(h/\Lambda) = t + \text{constant}$. In that case we can solve the equation as

$$t + \text{const} = \frac{1}{-\beta_1 \bar{g}^2} + \frac{\beta_2}{\beta_1^2} \log(1/\bar{g}^2) + o(\bar{g}^2)
 \tag{C.13}$$

We see that as we take \bar{g}^2 on a small circle around the origin $\bar{g}^2 \rightarrow \bar{g}^2 e^{2\pi i}$, then t performs a circle around infinity, but in addition we get a shift, $t \rightarrow t e^{-2\pi i} - \frac{\beta_2}{\beta_1^2} 2\pi i$. When we say that \mathcal{F} has a power series expansion in \bar{g}^2 we are saying that each term is invariant under this shift. However, t is not invariant and this one way to see that we need the logarithmic terms in (C.9), with coefficients determined by the lower order terms. All the logarithmic terms in (C.9) would vanish if β_2 were zero.

Now let us turn to the question of determining the leading logarithmic terms at each loop order in the $1/\sqrt{\lambda}$ expansion. One can determine such terms by performing manipulations similar to the ones performed above. However, it is also nice to see more directly how they are determined by using the renormalization group equations. For that purpose we

imagine that we compute the tree level and one loop expressions for \mathcal{F} using perturbation theory. The simplest answer is obtained by choosing $\mu = h$ in which case we get

$$f = h^2 \mathcal{F}, \quad \mathcal{F} = -\frac{1}{2} \left[\frac{1}{\bar{g}^2(h)} + a + o(\bar{g}^2) \right] \quad (\text{C.14})$$

where a is a constant that we will fix momentarily. We now run the coupling from the scale h to the scale 1 corresponding to the UV cutoff where the coupling is $\bar{g}_0^2 = \bar{g}^2(1) = \frac{2\pi}{\sqrt{\lambda}}(1 + \frac{\hat{c}}{\sqrt{\lambda}})$. The coupling $\bar{g}^2 1$ is the coupling run to the scale 1 in the $O(6)$ theory and constant \hat{c} is an unknown threshold correction. We can use the solution (C.13). We then see that we can express the coupling at scale h as

$$\frac{1}{\bar{g}^2(h)} = \frac{1}{\bar{g}_0^2} \left[1 + \bar{z} + \bar{g}_0^2 \frac{\beta_2}{\beta_1} \log(1 + \bar{z}) \right], \quad \bar{z} \equiv -\beta_1 \bar{g}_0^2 \log h \quad (\text{C.15})$$

where we neglect higher orders in \bar{g}_0^2 but kept all orders in \bar{z} . We now compute j to find

$$j = -\frac{\partial f}{\partial h} = h \frac{1}{\bar{g}_0^2} \left[1 + \bar{z} + \bar{g}_0^2 \frac{\beta_2}{\beta_1} \log(1 + \bar{z}) + \bar{g}_0^2 a - \bar{g}_0^2 \frac{\beta_1}{2} \right] \quad (\text{C.16})$$

We then solve for h as a function of $\bar{y} \equiv \bar{g}_0^2 j$. This gives

$$h = \frac{\bar{y}}{1 + \hat{z} + \bar{g}_0^2 \beta_1 \log(1 + \hat{z}) + \bar{g}_0^2 \frac{\beta_2}{\beta_1} \log(1 + \hat{z}) + a \bar{g}_0^2 - \bar{g}_0^2 \frac{\beta_1}{2}}, \quad \hat{z} \equiv -\beta_1 \bar{g}_0^2 \log \bar{y} \quad (\text{C.17})$$

The expression for the energy is then

$$\epsilon(y) = \frac{\bar{y}^2}{2\bar{g}_0^2} \left[\frac{1}{(1 + \hat{z})} - \frac{a \bar{g}_0^2 + \bar{g}_0^2 \beta_1 (1 + \frac{\beta_2}{\beta_1^2}) \log(1 + \hat{z})}{(1 + \hat{z})^2} \right] \quad (\text{C.18})$$

We now recall that $y = \frac{2\pi}{\sqrt{\lambda}} j$. Then $\bar{y} = y(1 + \frac{\hat{c}}{\sqrt{\lambda}})$. Similarly we can define $z = -\beta_1 \frac{2\pi}{\sqrt{\lambda}} \log y$, so that $\hat{z} = z(1 + \frac{\hat{c}}{\sqrt{\lambda}})$. We can then write (C.18) as

$$\epsilon(y) = \frac{y^2 \sqrt{\lambda}}{4\pi} \left[\frac{1}{(1 + z)} - \frac{k_3 \frac{2\pi}{\sqrt{\lambda}} + \frac{2\pi}{\sqrt{\lambda}} \beta_1 (1 + \frac{\beta_2}{\beta_1^2}) \log(1 + z)}{(1 + z)^2} \right] \quad (\text{C.19})$$

where k_3 is a combination of the unknown parameters. k_3 can be fixed by performing the one loop expansion of (C.19) and matching to the results in [11]. The one loop expansion of (C.19) is

$$\epsilon_{0+1} = y^2 \left[\frac{\sqrt{\lambda}}{4\pi} + \frac{\beta_1}{2} \log y - \frac{k_3}{2} \right] \quad (\text{C.20})$$

Comparing this to (3.10) we find that $-k_3/2 = 3/(4\pi)$. Inserting this in (C.18) and using the values of β_1, β_2 for the $O(6)$ model (C.4), we find

$$\epsilon = \frac{y^2 \sqrt{\lambda}}{4\pi} \left[\frac{1}{(1 + z)} + \frac{3}{\sqrt{\lambda}} \frac{1 + \log(1 + z)}{(1 + z)^2} \right], \quad z = \frac{4}{\sqrt{\lambda}} \log y \quad (\text{C.21})$$

The first term gives the leading log terms and the second gives the subleading ones that we had quoted in (3.13).

Let us now make some final remarks. Obviously, in the perturbative region we can express the free energy in terms of the running coupling constant $\bar{g}(\mu)$, at some scale μ

$$f(h) = h^2 \left(\frac{\mathcal{K}_1(h/\mu)}{\bar{g}^2(\mu)} + \mathcal{K}_2(h/\mu) + \mathcal{K}_3(h/\mu)\bar{g}^2(\mu) + \dots \right) \quad (\text{C.22})$$

Then the functions K_i are determined by the RG equations up to a constant.

The relation between the mass gap m and the scale defined in the \overline{MS} scheme via the equation (C.12), $\Lambda_{\overline{MS}}$, was computed in [13]

$$m = \left(\frac{8}{e} \right)^{\frac{1}{N-2}} \frac{1}{\Gamma(1 + \frac{1}{N-2})} \Lambda_{\overline{MS}} \quad (\text{C.23})$$

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